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The stability of an unsteady Kelvin–Helmholtz flow

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The investigation concerns the stability of an interface between two inviscid fluids of different densities which flow parallel to each other in an oscillatory manner. When the difference in the mean speeds is below the steady, critical speed for instability but is large compared to the amplitude of the fluctuations, parametric amplification of waves at the interface occurs, and the interface exhibits a resonance of a subharmonic nature. The paper concludes with a discussion of the stability of unsteady shear layers on the basis of the results.

1. Introduction

There would seem to be at least two reasons why the stability of time-dependent flows is of interest. First, one may be interested in how external effects which cause the basic flow to be unsteady but still laminar affect the stability of that flow. Thus, Donnelly (1964) has found experimentally that circular Couette flow can be stabilized somewhat by having the velocity of the inner wall oscillate about a mean value, while Gilbrech & Combs (1962) have obtained a similar result for pulsating flow through pipes. Some time ago, however, Fage (1938) found that axial flow through concentric pipes could be destabilized by having the inner pipe perform axial oscillations. A recent analytical investigation by Gershuni & Zhukhovitskii (1964) of the problem of convective instability of a fluid with a periodically varying temperature gradient indicates that the unsteadiness may have both stabilizing and destabilizing effects, depending upon the frequency and amplitude of modulation.

A second reason for interest in such flows is that, prior to its final breakdown into turbulence, a flow may develop from its steady, laminar form through one or more stages of finite-amplitude oscillation. Non-linear analyses based upon perturbing the primary, steady flow have provided information on these states of oscillation, but not on the final breakdown into turbulence. Hence it would seem worth while to take the view that the instability has grown to such a degree that the basic flow must be taken to be time-dependent and to perform a linear stability analysis based upon a model of this unsteady flow. Such an approach has been used by Greenspan & Benney (1963), who investigated by numerical techniques the stability of the time-dependent shear layer which is observed prior to the occurrence of bursts of turbulence in boundary-layer flow.

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The theory of parametric amplification, described, for instance, in the books by McLachlan (1947) and Stoker (1950), is important for the problem at hand and is associated with systems in which the parameters which define the natural frequencies when constant vary with time in a periodic manner. The behaviour of such systems is described by an equation of the Hill or Mathieu type. The stability diagram for the Mathieu equation

$$\frac{d^2 Y}{d\tau^2} + [\delta + \epsilon \cos \tau]\, Y = 0 \tag{1.1}$$

is shown in figure 1, and its characteristics lead us to expect, for instance, that an inverted pendulum can be stabilized by suitably oscillating it in the vertical direction, thus causing the effective gravitational force to vary periodically with time. More important to this paper is the fact that a normally stable pendulum can be made unstable by the vertical oscillations, especially if the frequency of

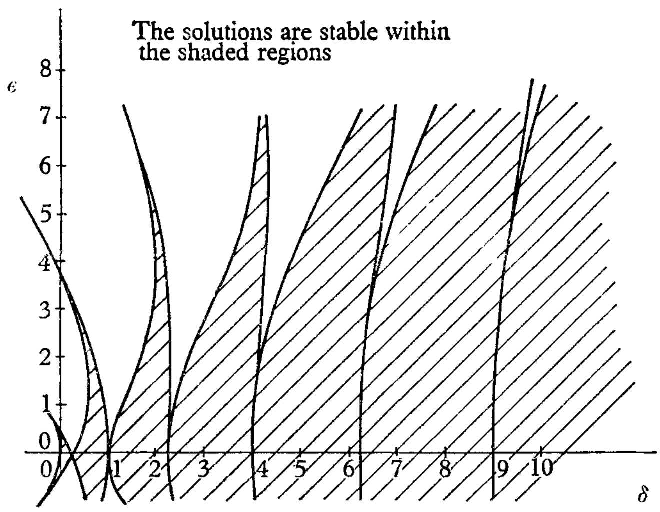

FIGURE 1. The stability boundaries for the Mathieu equation

$$\frac{d^2 Y}{d\tau^2} + [\delta + \epsilon \cos \tau]\, Y = 0.$$

vertical oscillation is exactly twice the frequency of the pendulum's natural motion (which corresponds to $\delta = \frac{1}{4}$ in figure 1). Some feeling for this subharmonic response can be gained by imagining the pendulum to be pulled upwards whenever the mass travels towards the position of equilibrium, thus increasing the kinetic energy of the mass, and pushed downwards whenever the mass travels away from the position of equilibrium, thus allowing the incremental increase of kinetic energy to be reflected in an increased amplitude of swing. As is clear from figure 1, instabilities are possible whenever $\delta = n^2/4$, n being any integer. However, dissipative effects tend to exert a strong influence on the cuspidal regions of instability for $n > 1$, and the subharmonic response is usually the only response to be observed experimentally.

Quite early in the development of fluid mechanics, Faraday (1831) noticed such a subharmonic response in the case of waves at an interface near a vibrating, elastic surface. More recently, Benjamin & Ursell (1954) applied the theory associated with the Mathieu equation to the problem of the generation of surface waves in a container which is partly filled with water and which oscillates vertically. They found excellent agreement between theory and experiment for the subharmonic response. Skalak & Yarymovych (1962) investigated the non-linear aspects of the development of the surface waves and found that further subharmonic responses become possible for sufficiently large amplitude oscillations of the container.

In this paper, the somewhat analogous problem of two fluids of different densities in parallel, time-dependent motion is considered. The corresponding steady problem is the classic one of Kelvin and Helmholtz. We show that the oscillations of the basic flow can cause a subharmonic resonance such that a wave which is neutrally stable in the absence of the oscillations and whose frequency is half of the frequency of the flow oscillations becomes unstable.

2. The model

We wish to consider the stability of a flow which is dependent upon time and which also satisfies the basic equations of motion. Consider the parallel flow of two fluids of different densities. Let x and y be co-ordinates in the plane of the interface, z the co-ordinate normal to the interface, u, v, w the corresponding velocity components, P the pressure, ρ the density, and g the gravitational acceleration. Let the velocities of the basic flow be $U_j(t)$, where $j = 1$ or 2 refers, respectively, to the lower or upper fluid. For an irrotational flow, an integral of the equations of motion may be written as

$$P_j/\rho_j + \tfrac{1}{2}(u_j^2 + v_j^2 + w_j^2) + gz + \partial\phi_j/\partial t = F_j(t), \tag{2.1}$$

where $F_j(t)$ is an arbitrary function of time and ϕ_j is a velocity potential.

In order to have a parallel basic flow, the interface must be plane, and the pressure must balance across it. By suitable choice of the $F_j(t)$, one can balance all terms except the terms involving the ϕ_j, which depend upon x as

$$\phi_j(x,t) = xU_j(t). \tag{2.2}$$

However, the difference in pressure can be set equal to zero if

$$\frac{dU_1}{dt} = \left(\frac{\rho_2}{\rho_1}\right)\frac{dU_2}{dt}, \tag{2.3}$$

which, for the statically stable case $\rho_2/\rho_1 < 1$, amounts to a balancing flow of smaller magnitude but in phase with the unsteady flow in the upper fluid. In reality, one would expect a modification of the flow in the lower fluid with a more complicated phase relationship due to viscous shearing action.

We have also assumed, in order to isolate the effects of flow unsteadiness, that the basic flow is spatially constant except for the discontinuous jump in density and velocity at the interface. For an analysis of the stability of a steady shear layer without stratification but with spatial variations, the reader is referred to a recent paper by Hocking (1964).

3. Stability analysis

We now consider the flow to be perturbed by a disturbance of sufficiently small magnitude so that we may consider the linearized version of (2.1) for the disturbance pressure, δp_j.

$$-\delta P_j = \rho_j U_j \delta u_j + g\rho_j \eta + \rho_j \frac{\partial \delta\phi_j}{\partial t}, \quad (3.1)$$

where η is the displacement of the interface from $z = 0$. We assume that the disturbance may be expressed in terms of its normal modes in the (x, y) directions and so express any function $f(x,y,z,t)$ as

$$f(x,y,z,t) = \bar{f}(z,t)\exp\{i(k_x x + k_y y)\}. \quad (3.2)$$

We take the fluid to be of infinite depth. The appropriate solutions of the equation of continuity are then

$$\delta\bar{\phi}_1 = A_1(t)\, e^{kz}, \quad \delta\bar{\phi}_2 = A_2(t)\, e^{-kz}, \quad (3.3)$$

where $k^2 = k_x^2 + k_y^2$. The equation which relates the displacement of the interface to the normal component of velocity is

$$\frac{\partial \eta}{\partial t} + U_j(t)\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial z}\delta\phi_j\bigg|_{z=0}. \quad (3.4)$$

Using this relationship, we may express the $A_j(t)$ in terms of $\bar{\eta}(t)$ as

$$A_j(t) = (-1)^{j+1}\left(\frac{1}{k}\right)\left(\frac{d\bar{\eta}}{dt} + ik_x U_j \bar{\eta}\right). \quad (3.5)$$

By use of (3.3) and (3.5), we may now write from (3.1) the jump in perturbation pressure across the interface as

$$\delta\bar{P}_2 - \delta\bar{P}_1 = \left(\frac{\rho_1+\rho_2}{k}\right)\frac{d^2\bar{\eta}}{dt^2} + 2i\left(\frac{k_x}{k}\right)(\rho_1 U_1 + \rho_2 U_2)\frac{d\bar{\eta}}{dt}$$
$$+\left\{g(\rho_1-\rho_2) - \frac{\rho_1 k_x^2}{k}U_1^2 - \frac{\rho_2 k_x^2}{k}U_2^2 + \frac{i\rho_1 k_x}{k}\frac{dU_1}{dt} + \frac{i\rho_2 k_x}{k}\frac{dU_2}{dt}\right\}\bar{\eta}. \quad (3.6)$$

A jump in the perturbation surface pressure is allowed due to the effect of surface tension so that

$$\delta\bar{P}_2 - \delta\bar{P}_1 = -k^2 T\bar{\eta}. \quad (3.7)$$

The final equation for $\bar{\eta}(t)$ is then

$$\frac{d^2\bar{\eta}}{dt^2} + 2ik_x(\alpha_1 U_1 + \alpha_2 U_2)\frac{d\bar{\eta}}{dt}$$
$$+\left\{gk(\alpha_1-\alpha_2) + \frac{k^3 T}{\rho_1+\rho_2} - \alpha_1 k_x^2 U_1^2 - \alpha_2 k_x^2 U_2^2 + i\alpha_1 k_x\frac{dU_1}{dt} + i\alpha_2 k_x \frac{dU_2}{dt}\right\}\bar{\eta} = 0, \quad (3.8)$$

where

$$\alpha_1 = \frac{\rho_1}{\rho_1+\rho_2}, \quad \alpha_2 = \frac{\rho_2}{\rho_1+\rho_2}. \quad (3.9)$$

It is convenient to eliminate the first derivative term by means of the substitution

$$\bar{\eta}(t) = \hat{\eta}(t)\exp\left\{-ik_x\int(\alpha_1 U_1+\alpha_2 U_2)\,dt\right\}. \tag{3.10}$$

The stability of $\bar{\eta}(t)$ is governed by that of $\hat{\eta}(t)$, and the equation for $\hat{\eta}(t)$ is then

$$\frac{d^2\hat{\eta}}{dt^2}+\left[gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\alpha_1\alpha_2 k_x^2(U_1-U_2)^2\right]\hat{\eta}=0. \tag{3.11}$$

When the velocities of the basic flow are constant, the interface becomes unstable when

$$\alpha_1\alpha_2 k_x^2(U_1-U_2)^2 \geqslant gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}. \tag{3.12}$$

The minimum value of $|U_1-U_2|$ for instability occurs when

$$k_x = k, \quad k^2 = g(\rho_1-\rho_2)/T$$

and is given by the formula of Kelvin

$$(U_1-U_2)_c^2 = \frac{2}{\alpha_1\alpha_2}\left[\frac{Tg(\alpha_1-\alpha_2)}{\rho_1+\rho_2}\right]^{\frac{1}{2}}. \tag{3.13}$$

The instability occurs because the pressure induced by the disturbance opposes the effects of gravity and surface tension and increases with the difference in velocities until the effective stiffness of the interface becomes nil. This effect is reflected in (3.11) by the change in the wave speed relative to the mean convective speed, the latter being expressed by the terms in the exponent of (3.10). From the preliminary remarks concerning parametric resonance, we suspect that a resonance might occur when the flow velocities vary periodically with time, and when, consequently, the wave speed is being modified continuously.

In accordance with the requirements concerning the basic flow, as discussed in §2, we consider the case when

$$U_2(t)-U_1(t) = \Delta U_0+\left(1-\frac{\alpha_2}{\alpha_1}\right)\tilde{U}\cos\omega t, \tag{3.14}$$

where ΔU_0 is the difference between the mean components and $\tilde{U}$ is the amplitude of the unsteady component of flow in the upper fluid. Then the governing equation (3.11) becomes

$$\frac{d^2\hat{\eta}}{dt^2}+\left[gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\alpha_1\alpha_2k_x^2(\Delta U_0)^2-\frac{\alpha_1\alpha_2}{2}\left(1-\frac{\alpha_2}{\alpha_1}\right)^2k_x^2\tilde{U}^2\right.$$
$$\left.-\alpha_1\alpha_2\left(1-\frac{\alpha_2}{\alpha_1}\right)k_x^2\left\{2\Delta U_0\tilde{U}\cos\omega t+\frac{1}{2}\left(1-\frac{\alpha_2}{\alpha_1}\right)\tilde{U}^2\cos 2\omega t\right\}\right]\hat{\eta}=0. \tag{3.15}$$

We first note that the mean effect of the oscillations is to augment the pressure, induced by the perturbation, which decreases the effective stiffness of the interface. Thus, in contrast to (3.12), the interface will have no effective stiffness when

$$\alpha_1\alpha_2k_x^2\left[(\Delta U_0)^2+\frac{1}{2}\left(1-\frac{\alpha_2}{\alpha_1}\right)^2\tilde{U}^2\right]\geqslant gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}. \tag{3.16}$$

Here we ignore the possibility that the system may be within one of the slender bands of stability which exist in figure 1 for $\delta < 0$. This result is due to the fact that the time-averaged kinetic energy for the unsteady flow (3.14) is greater than for the case of constant velocities. Because values of $\tilde{U}$ which are quite small in comparison with the mean velocity are mainly of interest, the modification of the classic result is not substantial in this respect. However, the situation is quite different when one considers the possibility of a subharmonic resonance.

The equation (3.15) has the form of Hill's equation. It is convenient to approximate the equation in two ways, depending upon whether

$$\Delta U_0 \gg \left|1-\frac{\alpha_2}{\alpha_1}\right|\tilde{U} \quad \text{or} \quad \Delta U_0 \ll \left|1-\frac{\alpha_2}{\alpha_1}\right|\tilde{U}.$$

We consider first the former case, which would seem to have the most interest.

$$\text{(i)}\ \Delta U_0 \gg \left|1-\frac{\alpha_2}{\alpha_1}\right|\tilde{U}.$$

The governing equation (3.15) is then approximated by the following equation:

$$\frac{d^2\hat{\eta}}{dt^2}+\left[gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\alpha_1\alpha_2k_x^2(\Delta U_0)^2 - 2\alpha_1\alpha_2k_x^2\left(1-\frac{\alpha_2}{\alpha_1}\right)\Delta U_0\tilde{U}\cos\omega t\right]\hat{\eta}=0, \tag{3.17}$$

which is of the Mathieu type. We note that the system is stable, as in the steady case, to perturbations transverse to the flow direction. Also, the oscillations have no effect when $\rho_1 = \rho_2$, in which case the oscillations are of the same amplitude and phase, and one can eliminate the explicit dependence on time from the beginning by simply defining a new variable of time. For this case, of course, the corresponding steady flow (for $T = 0$) is unstable for all wave-numbers.

As mentioned earlier, the subharmonic response is the most important because viscous effects would tend to eliminate the higher harmonic responses. The higher harmonic responses shown in figure 1 are all of order ϵ^2. For $|\epsilon| \ll 1$, the boundaries of the subharmonic region in figure 1 are defined approximately by

$$\delta = \tfrac{1}{4} \pm \tfrac{1}{2}\epsilon. \tag{3.18}$$

Contrasting (3.17) to (1.1), we may use (3.18) to write the approximate condition for instability in the present problem as

$$gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\alpha_1\alpha_2k_x^2(\Delta U_0)^2 = \frac{\omega^2}{4}\pm\alpha_1\alpha_2k_x^2\left(1-\frac{\alpha_2}{\alpha_1}\right)\Delta U_0\tilde{U}. \tag{3.19}$$

For values of $\tilde{U}$ such that these boundaries are surpassed, one solution of (3.17) will grow exponentially with time. Clearly, for small $\tilde{U}$, the instability is most likely when

$$\frac{\omega^2}{4} = gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\alpha_1\alpha_2k_x^2(\Delta U_0)^2. \tag{3.20}$$

Hence, given ω and ΔU_0, one can find a corresponding k to give the subharmonic response. Therefore the interface is always unstable under the present conditions. Assuming $k_x = k$, one can show that ω^2 increases monotonically with k up to

$$(\Delta U_0)^2 = \frac{1}{\alpha_1\alpha_2}\left[\frac{3g(\alpha_1-\alpha_2)T}{\rho_1+\rho_2}\right]^{\frac{1}{2}}. \tag{3.21}$$

For values of $(\Delta U_0)^2$ greater than (3.21) but less than (3.13), presumably three different values of k can satisfy (3.20) for given ω and ΔU_0. The growth rate would then determine which wave-number would be predominant. A schematic diagram of $(\omega^2/4)$ as a function of k for various ΔU_0 is shown in figure 2. For values

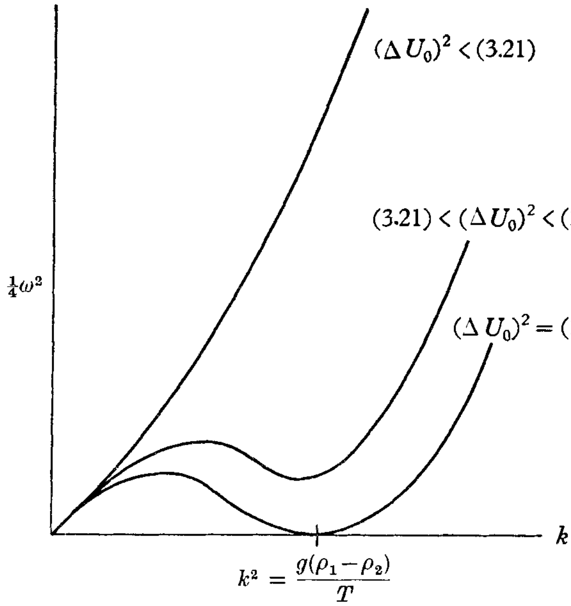

FIGURE 2. The relationship between frequency and wave-number for the subharmonic response.

of $(\Delta U_0)^2$ below the value given in (3.21), one can generally state that increasing ω corresponds to increasing k and that a disturbance of fixed wavelength will be excited by lower values of ω as ΔU_0 increases. The first of these statements is no longer valid when $(\Delta U_0)^2$ is in excess of the value given in (3.21).

It is known that, at $\delta = \frac{1}{4}$, the solutions of the Mathieu equation (1.1) will behave as $y(t) \sim y_0 \exp\{\pm \epsilon t/2\}$ for small ϵ. Thus when the corresponding condition (3.20) in the present problem is fulfilled, the interface will behave as $\hat{\eta}(t) \sim \hat{\eta}_0 \exp\{\pm qt\}$, where

$$q = \Delta U_0 \bar{U} \alpha_1 \alpha_2 \left(1 - \frac{\alpha_2}{\alpha_1}\right) k_x^2 \omega^{-\frac{1}{2}}. \tag{3.22}$$

and where ω is defined by (3.20). Hence, for constant ω and ΔU_0 and $k_x = k$, the growth rate is greater for disturbances with higher wave-numbers. Therefore, when $(\Delta U_0)^2$ is larger than the value given in (3.21) and three values of wave-number become possible for a given value of ω, the shortest wavelength disturbance will predominate. More generally, the growth rate per unit of time q increases monotonically with k up to the value of $(\Delta U_0)^2$ given in (3.21). Then, it also develops a relative maximum, as shown schematically in figure 3. The growth rate becomes singular when $(\Delta U_0)^2$ achieves the critical speed for Kelvin–Helmholtz instability because the perturbation technique which establishes the relation for the growth rate and which, with reference to (1.1), requires that $\delta \gg |\epsilon|$ is not valid as $\delta \to 0$ for non-zero ϵ. Finally, it is interesting to note that the growth rate per cycle of oscillation, i.e. q/ω, has a maximum for all ΔU_0 when $k^2 = g(\rho_1-\rho_2)/T$, i.e., for the disturbance with the minimum phase velocity in the absence of any flow.

These results are dependent upon the validity of the discontinuous profile model, however, and therefore are meaningful at most only for large wavelength disturbances.

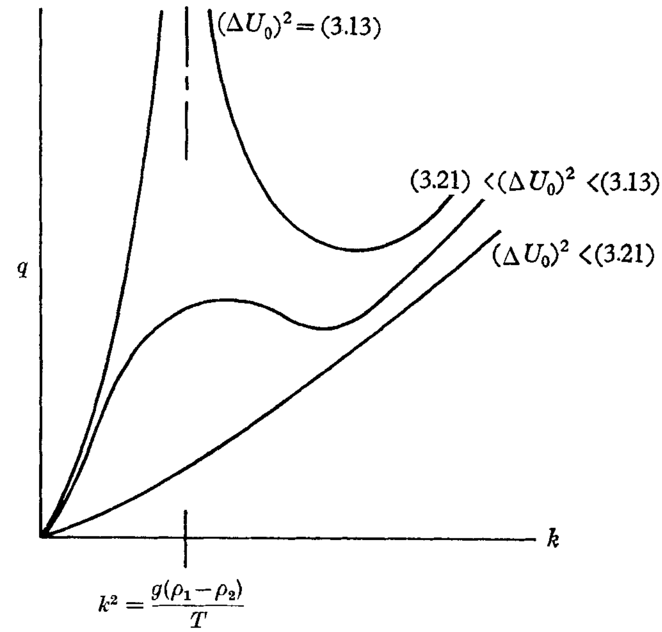

FIGURE 3. The relationship between growth rate and wave-number for the subharmonic response.

In order to understand the subharmonic resonance on a physical basis, it is perhaps helpful to move with the lower fluid and to consider the motion of a particle at the interface due to the influence of a neutrally stable wave. As the particle moves toward the equilibrium position, the in-phase restoring force will be greater than in the Kelvin–Helmholtz case if $|U_2-U_1| < |\Delta U_0|$ (see 3.11). On the other hand, if $|U_2-U_1| > |\Delta U_0|$ as the particle moves away from the equilibrium position, the restraining force will be relatively less, and the higher kinetic energy of the particle at the equilibrium position will result in an increased amplitude of departure away from this position. The mechanism is therefore quite similar to the motion of a pendulum due to a periodic vertical oscillation, which was discussed earlier.

$$\text{(ii)}\quad \Delta U_0 \ll \left|1-\frac{\alpha_2}{\alpha_1}\right| \tilde{U}.$$

The approximate form of (3.15) is then

$$\frac{d^2\hat{\eta}}{dt^2}+\left[gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\tfrac{1}{2}\alpha_1\alpha_2\left(1-\frac{\alpha_2}{\alpha_1}\right)^2 k_x^2\tilde{U}^2 -\tfrac{1}{2}\alpha_1\alpha_2\left(1-\frac{\alpha_2}{\alpha_1}\right)^2 k_x^2\tilde{U}^2\cos 2\omega t\right]\hat{\eta} = 0. \tag{3.23}$$

The stiffness of the interface will now only vanish when

$$\alpha_1\alpha_2k_x^2\tilde{U}^2 = 2\left(1-\frac{\alpha_2}{\alpha_1}\right)^{-2}\left[gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}\right], \tag{3.24}$$

which requires $\tilde{U}$ to be in excess of the Kelvin–Helmholtz result for ΔU_0 (3.12).

We also notice that the term explicitly dependent upon time has a frequency twice that of the imposed frequency. Hence the subharmonic response of (3.23) will actually be isochronous with respect to the flow oscillations.

The general form of (3.23) can be represented by the equation

$$\frac{d^2y}{dt^2}+[\delta'+\epsilon'+\epsilon'\cos 2\omega t]\,y = 0, \tag{3.25}$$

which may be placed in terms of (1.1) by defining

$$\tau = 2\omega t, \quad \frac{\delta'+\epsilon'}{4\omega^2} = \delta, \quad \frac{\epsilon'}{4\omega^2} = \epsilon. \tag{3.26}$$

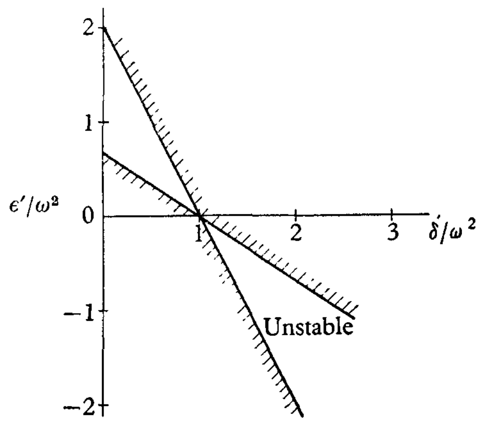

FIGURE 4. The subharmonic region of instability for the equation

$$\frac{d^2Y}{dt^2}+[\delta'+\epsilon'+\epsilon'\cos 2\omega t]\,y = 0.$$

If we assume that the stability of (3.25) can be decided in the same manner as the Mathieu equation, namely, by deciding how certain periodic solutions for $\epsilon = 0$ can be extended into the region $|\epsilon| > 0$, we may use (3.18) and (3.26) in in order to define the boundaries on the subharmonic region as

$$\delta' = \omega^2-\tfrac{1}{2}\epsilon', \quad \delta' = \omega^2-\tfrac{3}{2}\epsilon'. \tag{3.27}$$

The diagram is now asymmetrical with respect to the $\epsilon' = 0$ axis and is shown in figure 4. For our problem, $\epsilon' \leqslant 0$ and we consider only the lower half of figure 4. In contrast to the previous example, the most unstable condition no longer occurs for $\delta' = \omega^2$, which is stable for $|\epsilon'| > 0$ to this order of approximation. In fact, due to the shape of the diagram, instability will occur only for $\delta' > \omega^2$, thus requiring that

$$\omega^2 < gk\,(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}. \tag{3.28}$$

Then $\tilde{U}$ must fulfil a requirement with regard to its amplitude, namely,

$$\left|\tfrac{3}{4}\alpha_1\alpha_2\left(1-\frac{\alpha_2}{\alpha_1}\right)^2 k_x^2\tilde{U}^2\right| > \left|gk(\alpha_1-\alpha_2)+\frac{k^3T}{\rho_1+\rho_2}-\omega^2\right| > \left|\tfrac{1}{4}\alpha_1\alpha_2\left(1-\frac{\alpha_2}{\alpha_1}\right)^2 k_x^2\tilde{U}^2\right|. \tag{3.29}$$

In the previous case, when $\Delta U_0 \gg \tilde{U}_0$, instability was found to occur for any non-zero value of $\tilde{U}$ at the subharmonic condition. On this basis, the present case must be considered to be more stable.

4. Discussion of the stability of unsteady shear layers

The preceding section emphasized how an oscillating component of flow velocity, whose magnitude is quite small in comparison with the mean flow velocity, can destabilize that flow through the mechanism of parametric amplification. A similar result has been obtained for the flutter of a membrane in an oscillatory flow. There the unsteadiness also tends to destabilize the mode of instability intrinsic to the dissipative nature of the membrane and which has been discussed by Landahl (1962) and Benjamin (1963).

Not all flows, however, are as affected by the unsteadiness as are these models. For instance, the stability of the time-dependent, stratified shear flow

$$\left.\begin{array}{lll} z > 0, & U = az(1+\epsilon\cos\omega t), & \rho = \rho_2, \\ z < 0, & U = 0, & \rho = \rho_1, \end{array}\right\} \tag{4.1}$$

has also been investigated for $|\epsilon| \ll 1$ by Kelly (1964), who showed that no subharmonic resonance would occur. One should realize, however, that the corresponding steady flow ($\epsilon = 0$) has no instability at all. It is conjectured that unsteady effects are of most importance for those problems in which the corresponding steady flow is unstable for a certain range of parameters. Then instabilities might arise also outside this range for the unsteady problem if the disturbances in the steady problem have a non-zero wave velocity which is dependent upon wave-number outside this region. In other words, the system must be dispersive.

These last comments might be clarified by considering the behaviour of a discontinuous shear layer whose velocity varies linearly with height between two given values and whose thickness varies periodically with time. The numerical results of Greenspan & Benney (1963) indicate that neutrally stable disturbances, with wave-numbers in excess of the cut-off wave-number for the constant shear layer, can be excited in the unsteady case (see figure 2 of their paper). This result clearly indicates that the periodic contractions and expansions of the layer introduce a resonance of the Mathieu type. Analysis similar to that of the preceding sections reveals that one can relate the frequency of contraction to some wave-number, larger than the cut-off wave-number, through a subharmonic condition similar to that given in (3.20). This disturbance will undergo maximum amplification for a given frequency and for wave-numbers greater than the cut-off value, but waves close to it will also be amplified, depending upon the amplitude of oscillation (see 3.19). The growth rate of the instability decreases exponentially as the wave-number and frequency increase, so that very short wavelength disturbances are unlikely to be excited through this mechanism.

However, these results are dependent upon the assumption that the discontinuous profile yields physically meaningful results for wave-numbers in excess of the cut-off value. It is characteristic of the model that disturbances with wave-numbers in excess of the cut-off value are neutrally stable, and it is the interaction between these neutrally stable waves and the imposed unsteadiness which leads to parametric amplification. However, numerical calculations performed in connexion with the continuous profile $U(z) = \tanh z$, which is the type of flow intended to be simulated by use of the discontinuous shear profile, indicate that the disturbances are damped monotonically for wave-numbers in excess of the cut-off value. This is certainly true for finite Reynolds numbers, as Betchov & Szewczyk (1963) have shown, and is presumably true for the inviscid case. This extrapolation would be in agreement with Lin's (1955, chapter 8) interpretation of the correct inviscid limit for profiles with an inflexion point.

This conclusion indicates that seriously misleading results may arise for the case of an unsteady shear layer of the type considered by Greenspan & Benney, at least for wave-numbers beyond the cut-off value, through the use of a discontinuous profile. The unsteadiness will certainly not interact with a monotonically decaying, or growing, wave to the same degree as with a neutrally stable wave. This point can be appreciated by reference to figure 1 or by consideration of the effects of vertical oscillation on the motion of a heavily damped pendulum. The same criticism can be made of the use of a discontinuous profile to represent any antisymmetric flow. In the corresponding continuous case, Tatsumi & Gotoh (1960) have shown that the phase velocity does not vary with wave-number. Hence a co-ordinate system can be defined such that all disturbances with wave-numbers beyond the cut-off value will appear to be damped monotonically. By using a discontinuous profile, these disturbances would appear only to be neutrally stable. However, continuous profiles which are not antisymmetric may have a strong dependence of phase velocity on wave-number, as, for instance, in the case of a full jet profile. One might expect that unsteady effects could then cause disturbances with wave-numbers significantly beyond the cut-off value to become unstable.

Unsteadiness would otherwise seem to have importance only in possibly making a very limited range of wave-numbers near the cut-off value unstable and, as Greenspan & Benney show in their figure 5, in causing a shift to a somewhat higher value for the wave-number of the most unstable disturbance. This shift due to unsteady effects, however, is relatively small in comparison to the fundamental increase in wave-number due to the assumed formation of a very thin shear layer during the transition process. The very significant rise in wave-number and frequency due to the instability of this shear layer is, however, predictable on a quasi-steady basis. Hence it would seem that the unsteady nature of the shear layer is not of crucial importance to the results of Greenspan & Benney, as far as wave-number is concerned.

It is felt that the Kelvin–Helmholtz result remains valid, at least for long wavelength disturbances, in spite of the above objections. There the basic interaction is between the unsteadiness and a modified gravity wave which will damp out only in an oscillatory manner. Hence a first-order (ϵ) interaction is possible, although viscosity will cause somewhat higher amplitude oscillations to be required for resonance, just as the experimental results of Benjamin & Ursell (1954) reveal for their problem. The Kelvin–Helmholtz analysis indicates the importance of the basic system being dispersive, because, with reference to equations (3.10) and (3.11), it is clear that the unsteadiness interacts only with that part of the wave velocity which is dependent upon wave-number.

The above remarks must be regarded mainly as conjectures until an unsteady, continuous profile of the type suggested is investigated. It was, however, thought to be relevant to consider how unsteadiness affects the symmetric disturbance mode for a triangular jet, of thickness $2b(t)$, because this mode is neutrally stable for small values of wave-number as well as large. The discontinuous profile is more meaningful for the small wave-number régime, and one might expect the mode in the corresponding continuous case to exhibit a slightly damped, oscillatory behaviour. It should, however, be remembered that the antisymmetric mode is unstable for small wave-numbers. We take as the basic flow

$$\left.\begin{aligned} U(z,t) &= V(t)\left\{1-\frac{z}{b(t)}\right\}, \quad 0 \leqslant z \leqslant b(t), \\ U(z,t) &= V(t)\left\{1+\frac{z}{b(t)}\right\}, \quad -b(t) \leqslant z \leqslant 0, \\ U(z,t) &= 0, \qquad\qquad |z| \geqslant b(t). \end{aligned}\right\} \tag{4.2}$$

Both the thickness and velocity of the jet are taken to vary with time, as was assumed in the shear profile investigated by Greenspan & Benney. The equation for the disturbance stream function

$$\delta\psi = \overline{\psi}(z,t)\, e^{ikx} \tag{4.3}$$

is

$$\left(\frac{\partial}{\partial t}+ikU\right)\left(\frac{\partial^2}{\partial z^2}-k^2\right)\overline{\psi} = 0, \tag{4.4}$$

because $d^2U/dz^2 = 0$. The solution is therefore of the form

$$\left.\begin{aligned} z \geqslant b(t), &\quad \overline{\psi} = A(t)\, e^{-kz}, \\ 0 \leqslant z \leqslant b(t), &\quad \overline{\psi} = B(t)\, e^{kz} + C(t)\, e^{-kz}, \\ -b(t) \leqslant z \leqslant 0, &\quad \overline{\psi} = D(t)\, e^{kz} + F(t)\, e^{-kz}, \\ z \leqslant -b(t), &\quad \overline{\psi} = G(t)\, e^{kz}. \end{aligned}\right\} \tag{4.5}$$

The symmetric mode of the disturbance velocity $\delta u(x,z,t)$ is defined by taking

$$A = -G, \quad B = -F, \quad C = -D. \tag{4.6}$$

In order that the normal component of velocity be continuous at $z = b(t)$, 0, and $-b(t)$, we must have

$$\left.\begin{aligned} A &= B e^{2kb} + C, \\ B + C &= 0. \end{aligned}\right\} \tag{4.7}$$

Matching of the perturbation pressures at the same points gives the following equation for $B(t)$

$$2ib\,e^{2kb}\frac{dB}{d\tau}+\left\{1-e^{2kb}+2ikb\frac{db}{d\tau}e^{2kb}\right\}B=0. \tag{4.8}$$

Here a new variable of time,

$$\tau=\int V(t)\,dt, \tag{4.9}$$

has been introduced in order to eliminate the explicit dependence of the problem upon $V(t)$. The stability criterion remains the same as long as $V(t)$ is of one sign. The solution of (4.8) is simply

$$B=B_0\exp\left\{-ik\int\frac{1}{2kb}(1-e^{-2kb})\,d\tau-k(b-b_0)\right\}, \tag{4.10}$$

where the solution is written so that the disturbance has the phase velocity (cf. Rayleigh 1945, p. 395)

$$C_0=\frac{1}{2kb_0}(1-e^{-2kb_0}), \tag{4.11}$$

when $b(t)=b_0$, a constant. By use of (4.10), (4.7), and (4.5), we may write the following expressions for $\overline{\psi}(z,t)$:

$$\left.\begin{aligned} z\geqslant b(t),\quad \overline{\psi}(z,t)&=2B_0\exp\left\{-ik\int\frac{1}{2kb}(1-e^{-2kb})\,d\tau\right\}e^{-k(z-b_0)}\sinh kb,\\ 0\leqslant z\leqslant b(t),\quad \overline{\psi}(z,t)&=2B_0\exp\left\{-ik\int\frac{1}{2kb}(1-e^{-2kb})\,d\tau-k(b-b_0)\right\}\sinh kz.\end{aligned}\right\} \tag{4.12}$$

Hence the effect of a variation of $b(t)$ is simply to give rise to an apparent amplification or damping of the wave, depending on whether $z >$ or $< b(t)$ and $b(t) >$ or $< b_0$. A contraction of the jet will make the waves in the region $z < b(t)$ appear to become unstable, although the amplification will cease as soon as a constant value is reached. Damping will occur for the waves in the region $z > b(t)$. For the periodic case, the waves would be alternately damped and amplified with the same period as the thickness variation. While the result is interesting from an observational point of view, it is clear that no resonance mechanism of the type discussed earlier in the paper is involved for this mode.

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